Prerequisite Skills

This lesson requires the use of the following skills:

- evaluating expressions using the order of operations (5.OA.1)
- evaluating exponential expressions involving integer exponents (8.EE.1)
- factoring (8.EE.2)
- rewriting fractions in the simplest form (4.NF.1)
- adding like terms in an expression (8.EE.7b)

Introduction

You are already used to working with **whole numbers**, which are numbers that aren't fractions, decimals, or negative. That is, whole numbers are members of the set {0, 1, 2, 3, ...}. Some numbers that arise in the real world cannot be calculated by adding, subtracting, multiplying, or dividing whole numbers. Such numbers are known as irrational numbers. Irrational numbers can arise in many situations, but one common instance is when working with square roots. For example, suppose you wanted to make a triangular cabinet to fit in the corner of a room. If the sides of the cabinet that go along the wall are each 3 feet long, then the face of the cabinet that points toward the center of the room would be have to be $3\sqrt{2}$ feet long. In this lesson, we will investigate expressions involving square roots and determine whether the expression is an irrational number.

Key Concepts

- **Integers** are positive and negative whole numbers and 0.
- A **rational number** is a real number that can be written as $\frac{m}{n}$, where both m and n are integers and $n \neq 0$. That is, a rational number can be written as a fraction.
- All integers are rational numbers, because any integer a can be written as the fraction $\frac{a}{1}$.
- An **irrational number** is a real number that cannot be expressed as the ratio of two integers. In other words, it cannot be written as a fraction that has integers for both the numerator and denominator.
- **Real numbers** are the set of all rational and irrational numbers.
- The decimal representations of irrational numbers neither end nor repeat.

- The set of integers is closed under addition, subtraction, and multiplication, meaning that the sum or difference of any integers is an integer, and the product of any integers is also an integer.
- The set of rational numbers is closed under addition, subtraction, multiplication, and division, so the sum or difference of two rational numbers is rational and the product or quotient of two rational numbers is rational.
- On the other hand, the sum or difference of a rational number and an irrational number is an irrational number, as is the product or quotient of a rational number and an irrational number.
- An irrational number has a decimal that never ends or repeats. So, if you have a rational number and an irrational number, and you add, subtract, multiply, or divide them, the result will always be irrational because its decimal will never end or repeat.
- Take 2 and $\sqrt{2}$. The whole number 2 is rational because it can be written as a fraction: $\frac{2}{1}$. However, $\sqrt{2}$, which is approximately equal to 1.41421..., is irrational because its decimal never ends or repeats. The result of any operation performed on 2 and $\sqrt{2}$ will always be irrational. For example, adding them results in $2+\sqrt{2}\approx 2+2.41421\ldots\approx 3.41421\ldots$, and multiplying them results in $2\bullet\sqrt{2}\approx 2\bullet 1.41421\ldots\approx 2.82842\ldots$. Both 1.41421... and 2.82842... are irrational because their decimals never end or repeat.
- The sum, difference, product, or quotient of two irrational numbers can be either rational or irrational. To determine which it is, simplify the expression.
- A **radical expression** contains a square root, which can be shown using the radical symbol, $\sqrt{\ }$. The **square root** of a number *x* is a number that, when multiplied by itself, equals *x*. For example, 6 • 6 = 36, so 6 is a square root of 36.
- The number under the radical symbol is called the **radicand**.
- The square root of a number for which there is no rational root is an irrational number, such as $\sqrt{2}$.
- A **perfect square** is a number that is the square of a whole number. For example, 9 is a perfect square because $9 = 3^2$.
- A square root will be a rational number only if the radicand is a perfect square.
- A square root will be an irrational number only if the radicand is not a perfect square.
- An **irreducible radical** is a radical whose radicand contains no perfect square factors. In other words, the radical cannot be further reduced.

- To reduce a radical expression, factor out any perfect squares from the radicand, then use the product property of radicals, $\sqrt{m \bullet n} = \sqrt{m} \bullet \sqrt{n}$, to rewrite the expression as a product of the perfect squares and any remaining factors. Finally, evaluate the square roots of the perfect squares and simplify.
- Alternatively, rewrite the radicand in terms of its prime factorization, then use the product property to rewrite the expression as a product of factors with even powers and any remaining factors. Finally, evaluate the square roots of the factors with even powers and simplify.
- If there is a radical in the denominator of a simplified expression, rewrite it by multiplying the numerator and denominator by the radical in the denominator. This process is known as rationalizing the denominator.
- Two radical expressions containing the same irreducible radical are called like terms. For example, $2\sqrt{2}$ and $3\sqrt{2}$ are like terms because they both contain $\sqrt{2}$.
- A radical expression can be rewritten using the following properties.

Properties of Radicals

Property	Formula	Example
Addition property To add two like radical terms, add the coefficients.	$a\sqrt{m} + b\sqrt{m} = (a+b)\sqrt{m}$	$2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$
Product property The square root of a product is equal to the product of the square roots of the factors.	$\sqrt{m \bullet n} = \sqrt{m} \bullet \sqrt{n}$	$\sqrt{6} = \sqrt{2 \bullet 3} = \sqrt{2} \bullet \sqrt{3}$
Quotient property The square root of a quotient is equal to the quotient of the square roots of the dividend and the divisor.	$\sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$, $n \neq 0$	$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2} \bullet \sqrt{3}}{3} = \frac{\sqrt{6}}{3}$

(*continued*)

Property	Formula	Example
Power reduction property The square root of a number raised to an even power is equal to the number raised to half the original power.	$\sqrt{n^{2a}} = n^a$	$\sqrt{16} = \sqrt{2^4} = 2^2 = 4$
Rational denominator property To rationalize the denominator, multiply the numerator and denominator by the radical in the denominator.	$\frac{m}{\sqrt{n}} = \frac{m}{\sqrt{n}} \bullet \frac{\sqrt{n}}{\sqrt{n}} = \frac{m\sqrt{n}}{n}$	$\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2}$

Common Errors/Misconceptions

- incorrectly categorizing a root as irrational when an exact root exists
- incorrectly evaluating a root
- multiplying only the denominator by the radical when rationalizing a denominator
- incorrectly identifying a repeating or terminating decimal as irrational